where g^* is $g_2(x^*)$ and β is a constant. Guderley's solution corresponds to $\phi = g_1(x)f_2(y)$ with $\lambda = 18$, and describes both the supersonic flow that results from the expansion of a parallel sonic jet and the case of a subsonic jet which gradually changes into a sonic one. The function $f_2(y)$ discussed and graphed in Ref. 1, will not be reconsidered here.

Other solutions are possible; for example, g_2f_2, g_1f_1, and g_2f_1. The latter two are singular like y^{-2} on the axis and are probably of little physical interest. However, the solution $\phi = g_2(x)f_2(y)$ is meaningful and produces jets with alternative rates of expansion. Differentiation of Eq. (6) leads to $\mathrm{d}g/\mathrm{d}x = (3\beta + 3\lambda g^2/2)^{1/3}$. Let us introduce the nondimensionalizations $g(x) = A\tilde{g}(\tilde{x})$ and $x = C\tilde{x}$, where $C = 2^{1/2} |\beta|^{1/6} |\lambda|^{-1/2}/3^{1/3}$, and where $A = (2\beta/\lambda)^{1/2}$ if $\lambda\beta > 0$ and $A = (-2\beta/\lambda)^{1/2}$ if $\lambda\beta < 0$. Then, we are led to the following four allowable forms for $\mathrm{d}\tilde{g}/\mathrm{d}\tilde{x}$:

$$\mathrm{d}\tilde{g}/\mathrm{d}\tilde{x} = (1+\tilde{g}^2)^{1/3} > 0 \text{ if } \beta > 0, \quad \lambda > 0 \tag{7a}$$

$$= (1-\tilde{g}^2)^{1/3} \text{ if } \beta > 0, \quad \lambda < 0 \tag{7b}$$

$$= -(1-\tilde{g}^2)^{1/3} \text{ if } \beta < 0, \quad \lambda > 0 \tag{7c}$$

$$= -(1+\tilde{g}^2)^{1/3} < 0 \text{ if } \beta < 0, \quad \lambda < 0 \tag{7d}$$

Physically these solutions are easily interpreted. The horizontal velocity $\varphi_x = K/(\gamma+1) + g_2'(x)f_2(y)$ varies in the streamwise direction in a manner proportional to $\mathrm{d}\tilde{g}/\mathrm{d}\tilde{x}$. In the first case, Eq. (7a) gives a solution with ever-increasing $\tilde{g}(\tilde{x})$ in the positive x direction. Differentiation shows that $\mathrm{d}\tilde{g}'(\tilde{x})/\mathrm{d}\tilde{x} = 2\tilde{g}/(3(1+\tilde{g}^2)^{1/3})$. Thus, the factor $\mathrm{d}\tilde{g}/\mathrm{d}\tilde{x}$ in φ_x increases if $\tilde{g}$ is initially positive, so that the flow corresponds to an "accelerating" jet. If $\tilde{g}$ is initially negative, the rate of increase of $\tilde{g}'(\tilde{x})$ with respect to increases in $\tilde{x}$ is negative; the jet "decelerates," but since $\tilde{g}'(\tilde{x}) > 0$, it will accelerate once $\tilde{g}$ becomes positive. Similar considerations apply, obviously, to Eq. (7d).

Next consider Eq. (7b) for which $\mathrm{d}\tilde{g}/\mathrm{d}\tilde{x} = (1-\tilde{g}^2)^{1/3}$ and $\mathrm{d}\tilde{g}'(\tilde{x})/\mathrm{d}\tilde{x} = -2\tilde{g}/(3(1-\tilde{g}^2)^{1/3})$. The flow, therefore, accelerates or decelerates accordingly as $\tilde{g}'' > 0$ or $\tilde{g}'' < 0$. If $\tilde{g}$ is initially greater than 1, $\tilde{g}$ will decrease in the positive $\tilde{x}$ direction until $\tilde{g} \to 1+$. If $\tilde{g}$ is less than -1, $\tilde{g}$ will decrease and fall to $-\infty$. On the other hand, if $|\tilde{g}| < 1$ initially, $\tilde{g}$ will increase until $\tilde{g} \to 1-$, at which point $\tilde{g}$ will level off. As before, similar considerations apply to Eq. (7c). The axisymmetric form of Guderley's equation is also separable in the sense of this Note. The analogous $f_2(y)$ function is discussed in his monograph; the $g_2(x)$ function in that case is exactly the same as that treated in the forgoing analysis.

Closing Remarks

The preceding flows are obtained from a broad class of separable solutions and contain, as a special subset, Guderley's solution for the parallel sonic jet. They extend Guderley's results by allowing more general streamwise behavior, while at the same time retaining the same modal structure as given by his $f_2(y)$ crossjet function, and should be of considerable engineering interest. The solutions generated here should also provide useful check cases for numerical transonic algorithms.

References

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On Isotropic Two-Dimensional Turbulence

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IN 1935, Taylor,[1] from the basic equations of continuity and energy dissipation in three-dimensional space, derived time correlations between space rates of change of velocities as well as an expression for the time mean dissipation of trubulent energy. It is the purpose of this paper to derive parallel relationships for two-dimensional isotropic turbulence. Definite differences occur that lead to simple interpretation. Furthermore, the von Kármán equation for correlation of velocity components in various directions shows that the Taylor length scale, $L = \int_0^\infty R_y \mathrm{d}y$, is *always* zero for two-dimensional turbulence, a result which may throw doubt on the real significance of this definition.

Energy Dissipation

In x and y coordinates, with corresponding u and v velocity components, the equation for time-averaged energy dissipation per unit volume is:

$$W = \mu\left[2\overline{\left(\frac{\partial u}{\partial x}\right)^2} + 2\overline{\left(\frac{\partial v}{\partial y}\right)^2} + \overline{\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)^2}\right] \tag{1}$$

By isotropy,

$$\overline{\left(\frac{\partial u}{\partial x}\right)^2} = \overline{\left(\frac{\partial v}{\partial y}\right)^2} \text{ and } \overline{\left(\frac{\partial u}{\partial y}\right)^2} = \overline{\left(\frac{\partial v}{\partial x}\right)^2}$$

with which, upon expansion,

$$\frac{W}{\mu} = 4\overline{\left(\frac{\partial u}{\partial x}\right)^2} + 2\overline{\left(\frac{\partial u}{\partial y}\right)^2} + 2\overline{\left(\frac{\partial u}{\partial y} \cdot \frac{\partial v}{\partial x}\right)} \tag{2}$$

Hence, further reduction requires relations between mean squares and mean products.

Following Taylor precisely, there are only so many combinations of $\partial u/\partial x$, $\partial v/\partial x$, $\partial u/\partial y$, and $\partial v/\partial y$; namely,

$$\frac{n!}{2!(n-2)!} = \frac{4!}{2! \cdot 2!} = 6$$

With the four mean squares, the total number of mean products to reckon with is ten. Thus,

$$\overline{\left(\frac{\partial u}{\partial x}\right)^2}, \overline{\left(\frac{\partial v}{\partial x}\right)^2}, \overline{\left(\frac{\partial u}{\partial y}\right)^2}, \overline{\left(\frac{\partial v}{\partial y}\right)^2}, \overline{\frac{\partial u}{\partial x} \cdot \frac{\partial v}{\partial x}}, \overline{\frac{\partial u}{\partial x} \cdot \frac{\partial u}{\partial y}},$$

$$\overline{\frac{\partial u}{\partial x} \cdot \frac{\partial v}{\partial y}}, \overline{\frac{\partial v}{\partial x} \cdot \frac{\partial u}{\partial y}}, \overline{\frac{\partial v}{\partial x} \cdot \frac{\partial v}{\partial y}} \text{ and } \overline{\frac{\partial u}{\partial y} \cdot \frac{\partial v}{\partial y}}$$

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The number of terms can be reduced by isotropy, where the 1st = 4th, 2nd = 3rd, 5th = 10th, and 6th = 9th, leaving

$$\overline{\left(\frac{\partial u}{\partial x}\right)^2},\ \overline{\left(\frac{\partial u}{\partial y}\right)^2},\ \overline{\frac{\partial u}{\partial x}\cdot\frac{\partial u}{\partial y}},\ \overline{\frac{\partial u}{\partial x}\cdot\frac{\partial v}{\partial x}}\quad \overline{\frac{\partial u}{\partial x}\cdot\frac{\partial v}{\partial y}},\ \overline{\frac{\partial u}{\partial y}\cdot\frac{\partial v}{\partial x}}$$

to be related.

First, by continuity,

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0$$

whose mean square is:

$$\overline{\left(\frac{\partial u}{\partial x}\right)^2}+2\overline{\frac{\partial u}{\partial x}\cdot\frac{\partial v}{\partial y}}+\overline{\left(\frac{\partial v}{\partial y}\right)^2}=0$$

Therefore, since $\overline{(\partial u/\partial x)^2}=\overline{(\partial v/\partial y)^2}$,

$$\overline{\left(\frac{\partial u}{\partial x}\right)^2}=-\overline{\frac{\partial u}{\partial x}\cdot\frac{\partial v}{\partial y}} \tag{3}$$

whereas, in three-dimensional space

$$\overline{\left(\frac{\partial u}{\partial x}\right)^2}=-2\overline{\frac{\partial u}{\partial x}\cdot\frac{\partial v}{\partial y}}$$

Next, still following Taylor, it can be shown from rotation of the axes in an arbitrary direction that

$$\overline{\frac{\partial u}{\partial x}\cdot\frac{\partial u}{\partial y}}=\overline{\frac{\partial u}{\partial x}\cdot\frac{\partial v}{\partial x}}=0$$

and

$$\overline{\left(\frac{\partial u}{\partial x}\right)^2}=\overline{\left(\frac{\partial u}{\partial y}\right)^2}+\overline{\frac{\partial u}{\partial x}\cdot\frac{\partial v}{\partial y}}+\overline{\frac{\partial u}{\partial y}\cdot\frac{\partial v}{\partial x}} \tag{4}$$

Furthermore, volume integration of the two-dimensional dissipation equation yields

$$\overline{\left(\frac{\partial u}{\partial x}\right)^2}=-\overline{\frac{\partial u}{\partial y}\cdot\frac{\partial v}{\partial x}} \tag{5}$$

Insertion of Eqs. (3) and (5) in Eq. (4) then gives

$$\overline{\left(\frac{\partial u}{\partial x}\right)^2}=\frac{1}{3}\overline{\left(\frac{\partial u}{\partial y}\right)^2} \tag{6}$$

compared to

$$\overline{\left(\frac{\partial u}{\partial x}\right)^2}=\frac{1}{2}\overline{\left(\frac{\partial u}{\partial y}\right)^2}$$

in three dimensions.

Finally, Eqs. (5) and (6) in Eq. (2) yields

$$\frac{W}{\mu}=8\overline{\left(\frac{\partial u}{\partial x}\right)^2} \tag{7}$$

compared to 15 $\overline{(\partial u/\partial x)^2}$ in three dimensions and

$$\frac{W}{\mu}=\frac{8}{3}\overline{\left(\frac{\partial u}{\partial y}\right)^2} \tag{8}$$

compared to (15/2) $\overline{(\partial u/\partial y)^2}$ in three dimensions.

Correlation Relationships and Taylor Length Scales

von Kármán and Howarth,[2] in 1937, showed that, in general, the following equations hold in isotropic turbulence for correlation of x,y,z velocity components a distance r apart

$$R_{11}=\frac{\overline{u_1u_2}}{\overline{u_1^2}}=\frac{x^2(f-g)}{r^2}+g \tag{9}$$

$$R_{12}=\frac{\overline{v_1v_2}}{\overline{u_1^2}}=\frac{xy(f-g)}{r^2} \tag{10}$$

$$R_{13}=\frac{\overline{w_1w_2}}{\overline{u_1^2}}=\frac{xz(f-g)}{r^2} \tag{11}$$

where

$$r^2=x^2+y^2+z^2,\quad f=\frac{\overline{gg'}}{\overline{u_1^2}},\quad g=\frac{\overline{pp'}}{\overline{u_1^2}}$$

with g and p parallel and perpendicular to r, respectively.

Furthermore, they derived from continuity

$$\frac{\partial R_{11}}{\partial x}+\frac{\partial R_{12}}{\partial y}+\frac{\partial R_{13}}{\partial z}=0 \tag{12}$$

Substitution of Eqs. (9-11) in Eq. (12) yields

$$f=g-\frac{r}{2}\frac{\mathrm{d}f}{\mathrm{d}r} \tag{13}$$

However, for two-dimensional turbulence, substitution of Eqs. (9) and (10) in the first two terms of Eq. (12) results in

$$f=g-r\frac{\mathrm{d}f}{\mathrm{d}r} \tag{14}$$

compared to Eq. (13).

It is now interesting to compare the Taylor length scales for two- and three-dimensional turbulence.

Taylor's average size of the eddy was defined by him as:

$$L=L_g=\int_0^\infty R_y\mathrm{d}y=\int_0^\infty g\mathrm{d}r \tag{15}$$

Further, let

$$L_f=\int_0^\infty f\mathrm{d}r$$

Next, from Eq. (13),

$$2g=2f+r\frac{\mathrm{d}f}{\mathrm{d}r}=f+\frac{\mathrm{d}(fr)}{\mathrm{d}r}$$

integration of which gives

$$2\int_0^\infty g\mathrm{d}r=\int_0^\infty f\mathrm{d}r+\int_0^\infty \mathrm{d}(fr)$$

Hence, for three dimensions,

$$2L=L_f \text{ or } L=\tfrac{1}{2}L_f$$

Also, Taylor's minimum size eddy, λ, responsible for dissipation of energy in isotropic three dimensions is obtained by double differentiation of Eq. (13). Thus, $\lambda=\lambda_g=(1/\sqrt{2})\lambda_f$.

However, for two dimensions, integration of Eq. (14) gives

$$L=\int_0^\infty g\mathrm{d}r=\int_0^\infty \mathrm{d}(fr)$$
$$=0$$

Furthermore, for two dimensions, second differentiation of Eq. (14) produces

$$\lambda_g=(1/\sqrt{3})\lambda_f$$

The surprising result that L *always* equals zero in two-dimensional turbulence indicates that the Taylor definition of the averaged size of eddy [Eq. (15)] may not have real significance, certainly for two-dimensional turbulence. Apparently, L_f would be a more significant definition.

Loss of Kinetic Energy of Turbulence

The time rate of loss of turbulent energy per unit volume for two dimensions is:

$$-\frac{1}{2}\rho\frac{\mathrm{d}}{\mathrm{d}t}(\overline{u^2}+\overline{v^2})=-\rho\frac{\mathrm{d}\overline{u^2}}{\mathrm{d}t}=\frac{8}{3}\mu\overline{\left(\frac{\partial u}{\partial y}\right)^2} \tag{16}$$

Now, generally, according to Taylor,

$$\frac{\overline{u^2}}{\lambda_g{}^2}=\frac{1}{2}\overline{\left(\frac{\partial u}{\partial y}\right)^2}$$

Hence, from Eq. (16),

$$\frac{\mathrm{d}\overline{u^2}}{\mathrm{d}t}=-\frac{16}{3}\nu\frac{\overline{u^2}}{\lambda_g{}^2}$$

in two dimensions compared to

$$\frac{\mathrm{d}\overline{u^2}}{\mathrm{d}t}=-10\nu\frac{\overline{u^2}}{\lambda_g{}^2}$$

in three dimensions.

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Finite-Element Stress Analysis of Axisymmetric Bodies under Torsion

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Introduction

THE present study is motivated by the consideration of the stress analysis of artillery projectiles. During firing, the projectile is subjected to a combination of various loads which are: 1) axial load, due to linear acceleration of the projectile; 2) centrifugal load, due to angular rotation of the projectile; 3) torsional load, due to angular acceleration of the projectile; 4) internal load due to setback on H.E.; and 5) external load, due to gun tube constraint, band pressure, and balloting. In view of the complexity of the geometry of the projectile, a finite-element analysis must be performed in order to determine the stresses and deformations in the projectile. Since the projectile has an axis of rotational symmetry, it is only logical that an axisymmetric ring element would model it more accurately and efficiently. In an MIT study,[1] which was performed for AMMRC under contract, an axisymmetric ring element was developed based on the assumed-stress hybrid finite-element model. However, it can only treat axisymmetric loads of the projectile.

It is the goal of the present analysis to develop an axisymmetric solid-of-revolution element which can be used to determine the stresses and deformations in axisymmetric structural bodies under torsional loads. The assumed-stress hybrid model is employed to derive the element stiffness matrix such that the results can be combined with those from the MIT study.

Other finite-element formulations for solution of axisymmetric structural bodies under torsion can also be made. The axisymmetric quadrilateral element, based on the displacement formulation in the ANSYS finite-element program,[2] can be used for modeling axisymmetric structures with nonaxisymmetric loadings, such as bending, shear, or torsion. Different finite-element formulations have also been developed for the solution of torsion of nonprismatic bars.[3,4]

Formulation

The formulation of the element stiffness matrix is based on the assumed-hybrid model.[5] Since only structural problems in the shape of body of revolution are considered, an axisymmetric solid-of-revolution ring element is developed. The ring element has four modes and a general quadrilateral cross section. For convenience, the field equations in the cylindrical coordinates are used with u, v, and w denote, respectively, the components of displacement in the radial r, tangential θ, and axial z directions. In the solution of torsional problems, the semi-inverse method may be used and the components of displacement u and w are assumed to be zero. It can be shown that the solution obtained on the basis of such an assumption satisfies all the equations of elasticity and, therefore, represents the true solution of the problem.[6] Substituting $u=w=0$ in the six strain-displacement relationships and, making use of the fact that from symmetry the displacement v does not vary with the angle θ, one obtains:

$$\epsilon_r=\epsilon_\theta=\epsilon_z=\gamma_{rz}=0$$
$$\gamma_{r\theta}=\frac{\partial v}{\partial r}-\frac{v}{r},\quad \gamma_{\theta z}=\frac{\partial v}{\partial z} \tag{1}$$

From Eq. (1) and Hooke's law, it can be seen that of all the six stress components, only $\tau_{r\theta}$ and $\tau_{\theta z}$ are different from zero. As a result of this and the symmetry condition, two of the equilibrium equations are identically satisfied and the third one becomes:

$$\frac{\partial\tau_{r\theta}}{\partial r}+\frac{\partial\tau_{\theta z}}{\partial z}+\frac{2\tau_{r\theta}}{r}=0 \tag{2}$$

Hence, in the subsequent formulation of an axisymmetric solid-of-revolution ring element, one is only concerned with the displacement v and the equilibrium equation given by Eq. (2).

The assumed-stress hybrid model is based on a modified complementary energy principle. It assumes compatible displacements along the interelement boundaries and a stress

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